Nonlinear correlations in stock time series and market prices of options

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Abstract

Linear and nonlinear correlations between price changes of the German stock index DAX and the implied volatility index VDAX on a daily basis, January 92 – September 98, are studied.

A rescaling of the underlying stock index changes by the implied volatility reduces the non-Gaussian nature of the probability distribution of these price changes. The nonlinear time correlations are reduced as well.

Key words: implied volatility; nonlinear correlations; non-Gaussian distributions
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1 Introduction

It was already observed in the financial literature of the early sixties, notably the pioneering work of Mandelbrot [3], that stock price changes have non-Gaussian properties, e.g. fat tails [3–5]. In the last few years physicists started to focus their interest on the nontrivial scaling properties of these probability density functions [6–13] as well as on nonlinear correlations in financial time series [12,13]. It is well known that in liquid markets the autocorrelation function of price changes rapidly goes to zero within a few minutes. But this does not imply the independence of the price changes. As an example the autocorrelation function of the squared price changes decays slowly in the order up to a month. In the financial community this is known as the volatility clustering effect.

There are a large number of models trying to explain the upper two empirical facts. The mostly used models are autoregressive conditional heteroscedasticity (ARCH) models [14], their generalisations (GARCH) [15] and stochastic volatility (SV) models [16].

Much effort was spent in improving these kind of models for using them for option pricing in the recent past [17].

However furthermore it will be argued that most of the non-Gaussian properties are already included in the market prices of options.

2 Brownian motion and and non constant volatility

The basic assumption in the Black & Scholes option pricing theory [18] is that the price of the underlying asset $S(t)$ is described by a geometric Brownian motion

$$\dot{S}(t) = S(t) \cdot (\mu + \sigma \cdot \xi(t)) \tag{1}$$

where $\xi(t)$ is a Gaussian white noise, μ is a deterministic function and the volatility σ is a constant. To compare empirical data with the assumptions of Equ. (1) one defines the return as

$$r_{\Delta t}(t) = \ln(S(t)) - \ln(S(t - \Delta t)) \tag{2}$$

where Δt is the time between two measurements of the price. For this studies, done in this paper, the daily data of the German stock index DAX [1] of 1687 trading days from January 1992 to September 1998, e.g. $\Delta t = 1 \text{ trading day}$, were used.

In Fig. 1 the probability density function of the daily returns is plotted. Though, there are deviations from a Gaussian distribution, they are small but notable even for the daily data. Using high frequent time series one finds much larger differences to a Gaussian distribution [7]. More remarkable deviations from Gaussian noise are found in the nonlinear autocorrelation functions. As an example the autocorrelation function of the squared returns

$$K^{22}(\tau) = \frac{\langle (r(t+\tau) - \bar{r})^2 \cdot (r(t) - \bar{r})^2 \rangle}{\langle (r(t+\tau) - \bar{r})^2 \rangle^2} \tag{3}$$

where $\bar{r} = \langle r(t) \rangle$ and the angular brackets denote the time average is plotted in Fig. 2. According to Wick's theorem for Gaussian noise the nonlinear autocorrelation functions should obey $K^{22}(0) = 3$ and $K^{22}(\tau \neq 0) = 1$. However the empirical function drops down slowly from $K^{22}(0) = 7.3$ to one. This is in contradiction to the constant volatility assumption in Equ. (1).

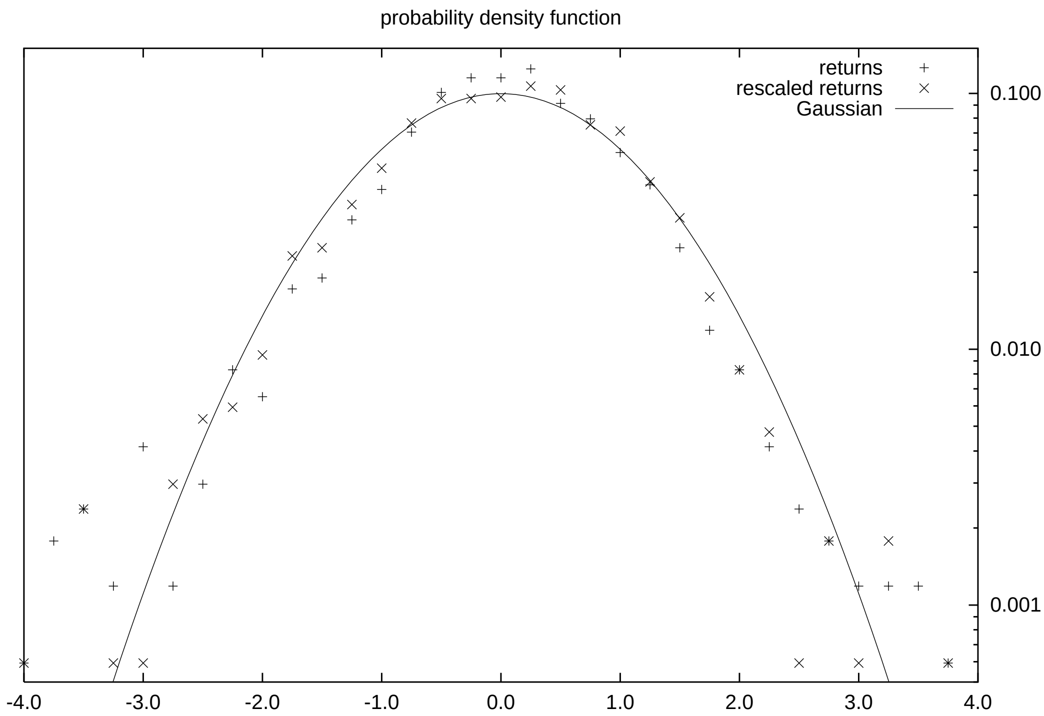

Fig. 1. Probability density of daily returns of the DAX index (pluses) and of the rescaled returns (crosses). The solid curve is a Gaussian. All distributions are normalised to zero mean and variance one.

3 Implied volatility as an estimator for the expected price changes

Although one of the basic assumptions of the Black & Scholes option pricing theory is not valid, the Black & Scholes formula is used in practise 'backwards' in order to calculate the so-called implied volatility [19]. This is the volatility one has to insert into the Black & Scholes formula to obtain the observed market price for a certain option. The implied volatility is hence the market consensus of volatility for a certain time horizon (time until the expiration of the option) and for a certain strike price of the option.

Now it will be tested whether in the market prices of options the non-Gaussian properties of the underlying are already included or not. Therefore we rescale the returns of the underlying of the option by the implied volatility of at-the-money options $\sigma_{impl}(t)$, e.g.

$$r_{res}(t) = \frac{r(t)}{\sigma_{impl}(t)} \tag{4}$$

If the implied volatility $\sigma_{impl}(t)$ would match with the time dependent volatility σ from Equ. (1), the rescaled return would be a measurement for the noise $\xi(t)$ and hopefully it would be Gaussian.

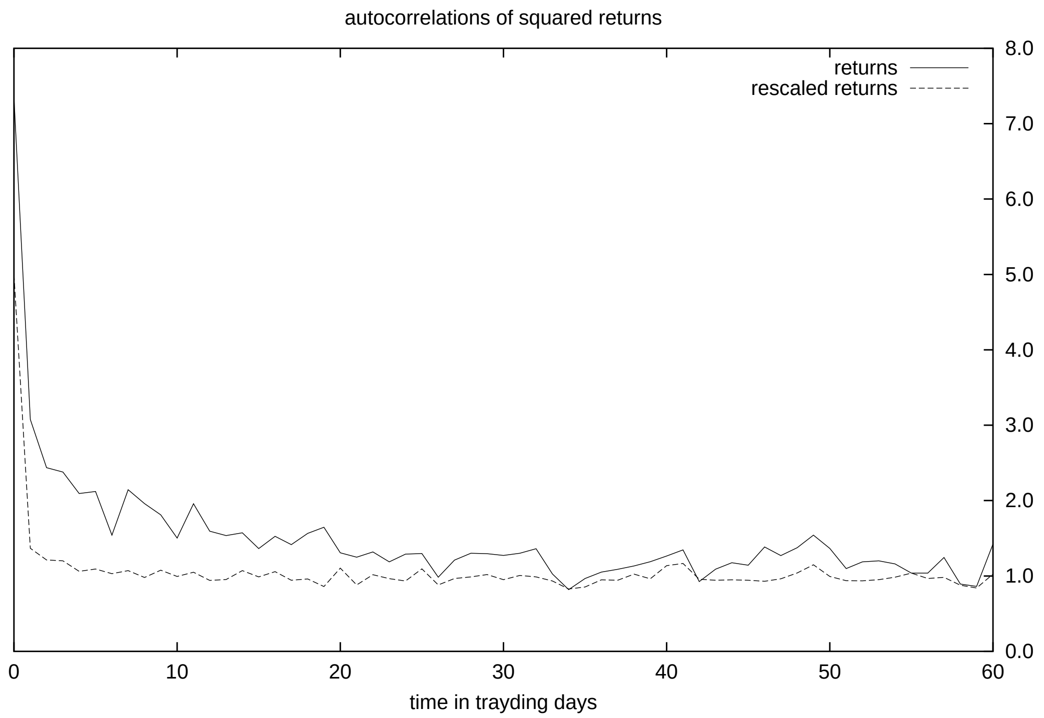

Fig. 2. Autocorrelation function of the squared returns $K^{22}(\tau)$ (solid line) and of the squared rescaled returns $K^{22}_{res}(\tau)$ (dashed lined). For white Gaussian noise it should be constantly one ($\tau > 0$)

Instead of the prices of the options on the DAX the German volatility index VDAX [2] is directly used as $\sigma_{impl}(t)$ in order to rescale the daily returns of the DAX. Since the VDAX is calculated for a remaining lifetime of 45 calendar days one can assume that even allowing a time varying volatility $\sigma(t)$ in Equ. (1) the resulting returns for 45 days would be drawn from a Gaussian distribution. Hence the squared implied volatility should coincide with the expectation value of the time average of the squared volatilities

$$\sigma_{impl}(t)^2 \approx E\left[\frac{1}{T}\int_{t}^{t+T} \sigma(t')^2 \,\mathrm{d}t'\right] \tag{5}$$

where T is the lifetime of the option, e.g. $T = 45$ calendar days.

Due to (5) it can not be expected that rescaling by the implied volatility will remove all non-Gaussian properties. However the probability density function of the rescaled returns is slightly better fitted by a Gaussian distribution then the unscaled (Fig. 1). The kurtosis as a measure for non-Gaussian behaviour is significant smaller: $\kappa_{res} = 2.0 < \kappa = 4.3$. Furthermore the nonlinear autocorrelation function $K^{22}_{res}(\tau)$ is much faster decaying as depicted in Fig. 2.

Table 1
Fitted parameters of the model (6,7).

μ	=	$6.2 \cdot 10^{-4}$	$\langle \xi_S(t)\xi_S(t') \rangle$	=	$3.4 \cdot 10^{-7}\delta(t,t')$
κ	=	0.014	$\langle \xi_\sigma(t)^2 \rangle$	=	$2.0 \cdot 10^{-3}$
Θ	=	2.83	$\langle \xi_\sigma(t)\xi_\sigma(t-1) \rangle$	=	-0.1 $\cdot \langle \xi_\sigma(t)^2 \rangle$
			$\langle \xi_S(t)\xi_\sigma(t) \rangle$	=	$-0.49\sqrt{\langle \xi_S(t)^2 \rangle \langle \xi_\sigma(t)^2 \rangle}$

4 Discussion

Looking at the upper analysis it seems that the non-Gaussian properties of the underlying (DAX) are already priced in the market prices of the options. To go a little bit more into detail the following model was fitted to the two dimensional time series of the underlying (DAX) and of the implied volatility (VDAX)

$$\dot{S}(t) = S(t) \cdot (\mu + \sigma(t) \cdot \xi_S(t)) \quad (6)$$
$$\dot{\sigma}(t) = \sigma(t) \cdot (\kappa \cdot (\Theta - \ln \sigma(t)) + \xi_\sigma(t)) \quad (7)$$

with the parameters according to Tab. 4. $\xi_S(t)$ and $\xi_\sigma(t)$ are negativ correlated noise, fairely approximated by a Gaussian ($\xi_S(t)$ more then $\xi_\sigma(t)$).

This model, which is quite similar to the stochastic volatility models used for option pricing [16], is able to reproduce these two upper quoted empirical facts. Due to the negative correlation of the two types of noise the resulting probability distribution of the underlying returns will be non-Gaussian, at least on short time scales. Furthermorte the slow decay of the nonlinear correlation functions follows from the long memory and mean reverting properties of the volatility (deterministic term in Equ. (7)). The parameter κ can be interpreted as an inverse decay time of volatility shocks with a half-life period of around 48 trading days, which is rather long and can be compared with the characteristic timescale of Fig. 2.

5 Conclusion

It seems to be clear that the market consensus is much better in pricing options than any theory can do.

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