# Historical Volatility Distribution in Gaussian and GARCH(1,1) models

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**Abstract**

On experimental data the historical volatility is usually calculated by averaging the local variance (or its generalizations) over a finite time window. Already in the case of a constant volatility in the Gaussian model the resulting historical volatility is non-Gaussian distributed. We will calculate historical volatility distributions in the Gaussian and GARCH(1,1) model for different time window size and compare them with those obtained from the S&P500 data [1, 2, 3].

# Native Stochastic Volatility

Historical volatility $\sigma_N$ is usually measured by calculating the standard deviation of the logarithmic price increments (returns)

$$r_{i=t/\Delta t} = G_{\Delta t}(t) = \ln S(t) - \ln S(t - \Delta t) \qquad (1)$$

for a time window $N\Delta t$, where $\Delta t$ is the time scale. To compare with [1, 2, 3] we choose $\Delta t = 30$ or 1 minute. The measurement of the standard deviation

$$\sigma_N = \sqrt{\frac{1}{N(N-1)} \left( N \sum_{i=1}^{N} r_i^2 - \left( \sum_{i=1}^{N} r_i \right)^2 \right)} \qquad (2)$$

is due to the sampling stochastic, already in a constant volatility world.

# A Constant Volatility World

Assume a market that is normally distributed (in its returns) with constant volatility $\sigma_\infty$. From sampling theory its known that

$$x = (N-1)\frac{\sigma_N^2}{\sigma_\infty^2} \tag{3}$$

is $\chi^2$-distributed with $N-1$ degrees of freedom. Therefor the $N$-period historical volatility distribution can be calculated as

$$\begin{aligned} P(\sigma_N) &= \chi^2_{N-1}(x)\frac{\mathrm{d}x}{\mathrm{d}\sigma_N} && (4) \\ &= \chi^2_{N-1}\left((N-1)\frac{\sigma_N^2}{\sigma_\infty^2}\right) 2(N-1)\frac{\sigma_N}{\sigma_\infty^2} && (5) \end{aligned}$$

$$= \quad \frac{n^{n/2}\sigma_N^{n-1}\exp\left(-\frac{n\sigma_N^2}{2\sigma_\infty}\right)}{2^{n/2-1}\sigma_\infty^n\Gamma(n/2)} \quad ; \quad n = N-1 \quad . \tag{6}$$

## A GARCH(1,1) World

In [5] the following GARCH(1,1) was compared to a S&P500 analysis

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1} + \beta_1 \sigma_{t-1}^2 \quad , \tag{7}$$

where $\alpha_0, \alpha_1, \beta_1$ are control parameters and $r_{t-1}$ is a random variable with a Gaussian distribution of zero mean and standard deviation $\sigma_{t-1}$. The choice of parameters $\alpha_0 = 2.3 \times 10^{-5}$, $\alpha_1 = 0.09105$ and $\beta_1 = 0.9$ leads to the same unconditional variance

$$\sigma_\infty^2 = \frac{\alpha_0}{1-\alpha_1-\beta_1} \approx 2.57 \times 10^{-3} \tag{8}$$

as observed in the empirical analysis of the S&P500 changes for the time intervals $\Delta t = 1$min. To compare with the S&P500 data [1, 2, 3] equation (8) is iterated with a time lag $\Delta t = 1$min and integrated to $\Delta t = 30$min.

In figure 1 the (historical) volatility distribution for different time windows is plotted. For comparison the distribution for the constant Gaussian model is shown also. Figure 3 shows in a log–log–scale the powerlaw behavior for extreme volatilities. The results are in a good agreement with those obtained from the S&P500 as depicted in figure 2.

## Detrended Fluctuation Analysis

To compare with [2, 3] we first define the displacement $y(t)$ by integrating $|\sigma_t|$ on a time scale $\Delta t = 30$min

$$y(t) = \sum_{i=1}^{t} |\sigma_i| \tag{9}$$

and calculate the mean square fluctuation $F(t)$ around the average displacement given by:

$$F(t) = \sqrt{\langle(\Delta y(t))^2\rangle - \langle\Delta y(t)\rangle^2} \quad , \tag{10}$$

where $\Delta y(t) = y(t+t_0) - y(t_0)$ and $\langle\ldots\rangle$ is the average over all initial times $t_0$.

In the Gaussian model $F(t)$ follows the scaling law $F(t) \propto t^{0.5}$. In the GARCH(1,1) process the scaling exponent is approximately identical with this of the S&P500 data (see figures 4 and 5). However for a long time horizon the scaling breaks down to the scaling of the Gaussian process.

## Discussion

In summary, the GARCH(1,1) process describes adequate the scaling properties of the S&P500 volatility distribution however it fails to describe properly the scaling of the distribution of price changes itself [5].

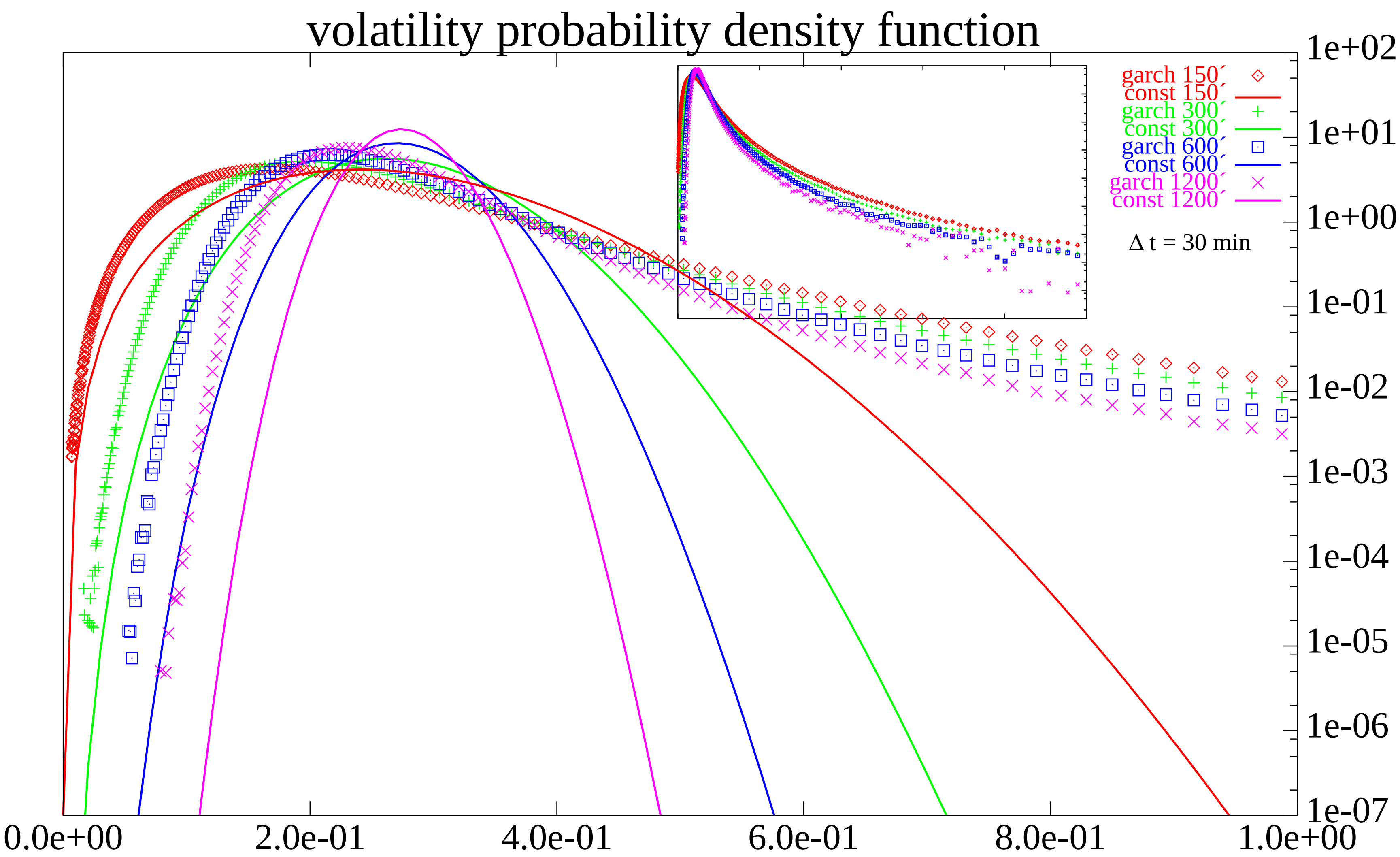


Figure 1: Volatility distribution in a constant volatility and GRACH(1,1) world. The time lag $\Delta t = 30$ min. as in [1, 2, 3]. The standard deviations are calculated in a time window of 150, 300, 600 and 1200 minutes.

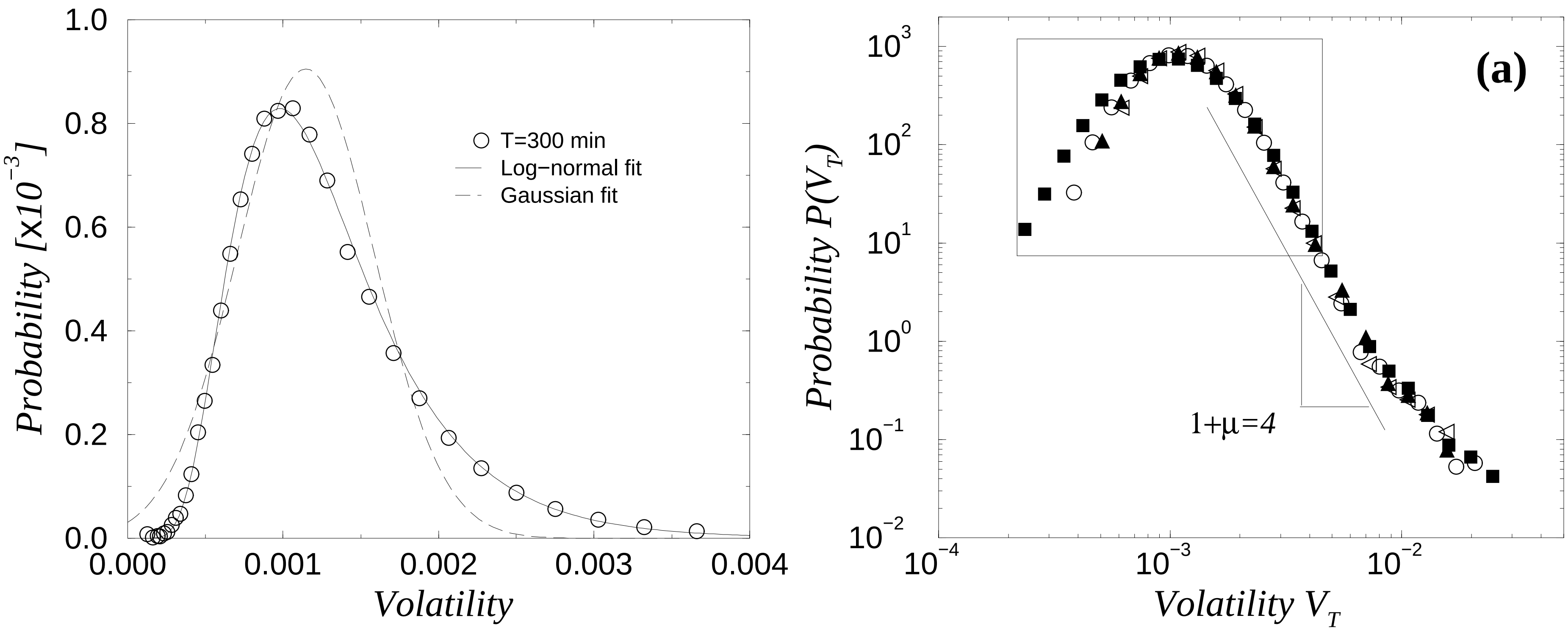


Figure 2: Stolen from [3]. Volatility distribution for the S&P500 data with different time windows of 120, 300, 600, 900 minutes ($\Delta t = 30$min).

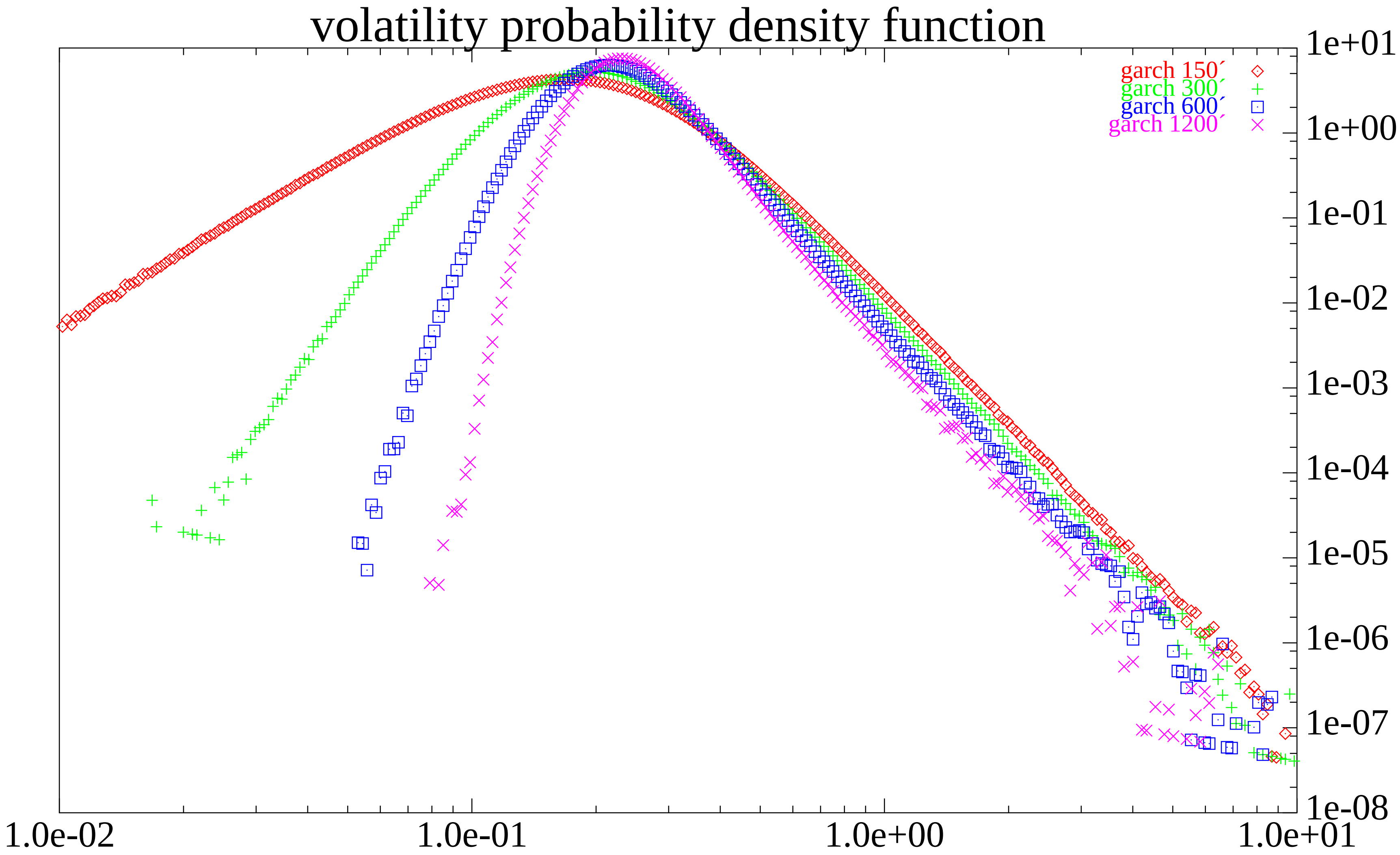


Figure 3: Same as in Fig. 1 for the GARCH(1,1) world in a log–log–scale. Note the powerlaw for extreme volatilities.

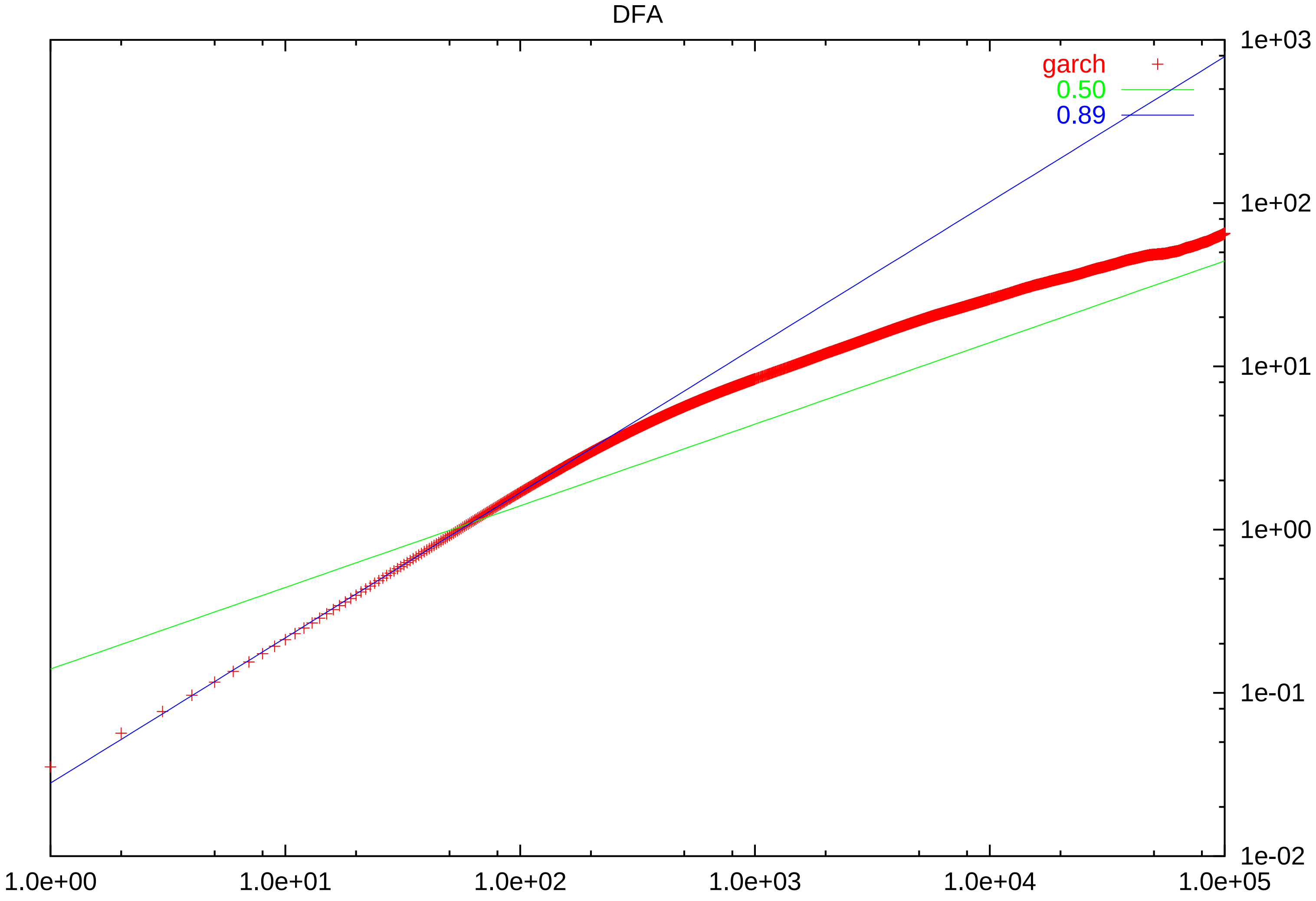


Figure 4: Detrended fluctuation analysis for the GARCH(1,1) process.

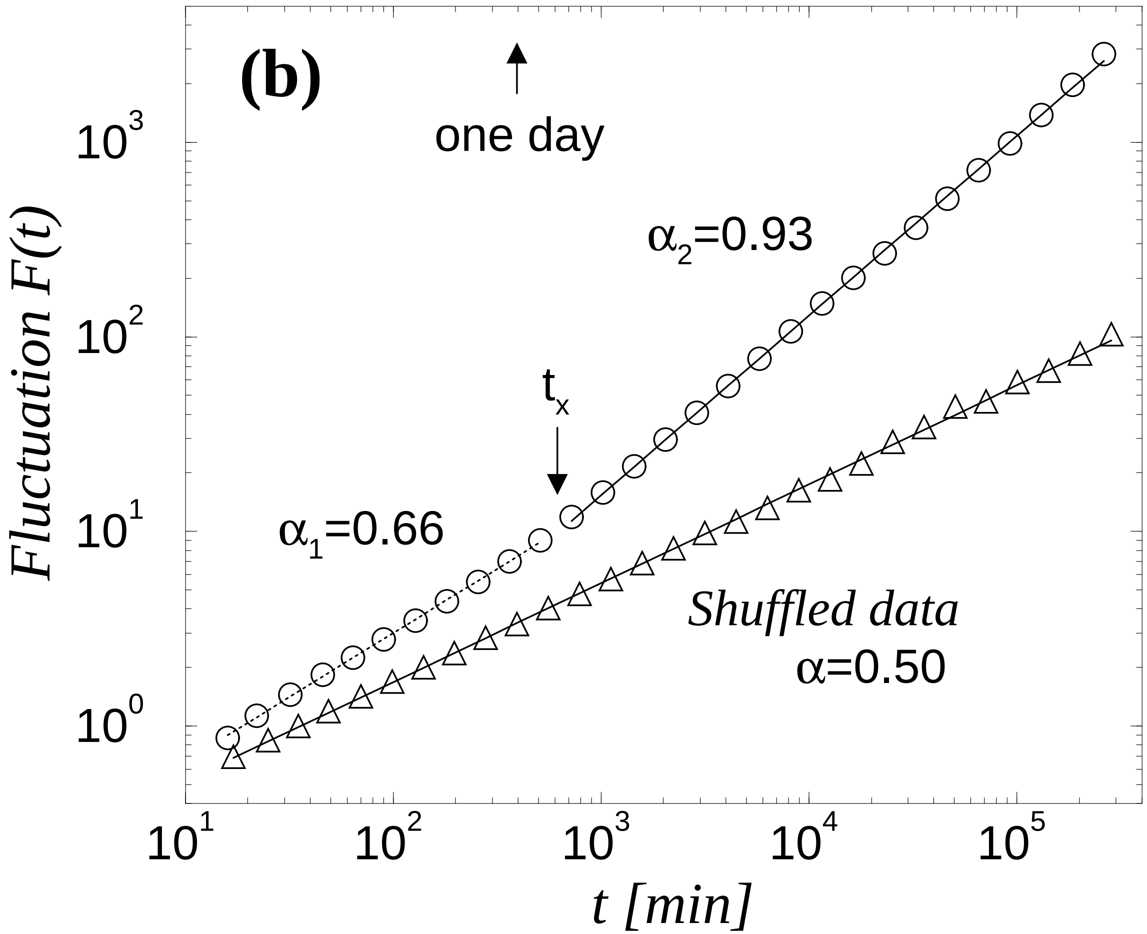


Figure 5: Stolen from [3]. Detrended fluctuation analysis for the S&P500 data.

## References


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